“Temporary General Equilibrium in a Monetary Economy with Multi-planning Periods,” in Essays and Studies Published in Commemoration of the 20th Anniversary of Otemon-gakuin University, Faculty of Economics, in Japanese, (March 1987).

On 10th, May, 2021, this paper was translated by Kazushi Nishimura in the following.

Temporary General Equilibrium in a Monetary Economy with Multi-planning Periods

Kazushi Nishimura

1. Introduction

In a monetary economy, a series of studies based on the temporary general equilibrium theory has been actively researched from various angles since 1970s. It shows certain results by Grandmont’s book [9]. However, most of setting of models are for two periods. When models are extended for more than three periods, it is not shown how to solve various problems in multi-planning periods as Grandmont himself said that the solutions were not evident. Among them, there was only one work of Chetty and Dasgupta [2] but it does not construct the multi-planning period’s theory based on essence of Grandmont’s theory. Especially, their paper did not explicitly grasp the preferences theory by Grandmont.

Hence I proved the existence of von Neumann-Morgenstern utility function defined by multi-variate utility function because I had extended the Grandmont’s preference theory to the multi-planning period’s one. In this paper, I apply the multi-variate von Neumann-Morgenstern utility function to temporary general equilibrium theory in a monetary economy with multi-planning period’s.

Furthermore, when the Grandmont’s theory [12, 13] seemed the extension of Patinkin’s theory [23], in the monetary economy with multi-planning periods, a consumer’s preferences to multi- period’s consumption must naturally enter into the theory as the Fisherian time preference theory. Under a certain economy, usually, we suppose that the utility of each period’s consumption is able to compare in time with the time discount element. Then, total utility over multi- periods is denoted by the summation of each period’s utility. Under the uncertainty, since there exists a linear utility, it can be proved that there exists a comparable time preference utility function within less axioms [21, 22].

Next, in Grandmond’s temporary general equilibrium theory, a von Neumann-

Morgenstern utility function is not used directly, but transformed to the indirect utility of financial assets on which Patinkin insisted, by combined the von Neumann-Morgenstern utility function with his Grandmond's price expectation function. Therefore, the price expectation formation occupies theoretically, the important position in Grandmond's temporary general equilibrium theory. Furthermore, in Green's studies [14, 15] that treated the temporary general equilibrium theory with forward markets, and the problem of bankruptcy after Green's studies, the condition of Grandmond's price expectation function is sufficient for the existence of market equilibrium. In Hool [19,20], the tight of the price expectation function is replaced with Clower [3, 4]condition 'money is used as exchange means.'

In short, since multi-periods model has not thoroughly studied in the Grandmond's temporary general equilibrium theory, I explain the time preference theory under uncertainty with the multi-variate von Neumann-Morgenstern utility function, introduce the time structure of financial assets and show the solution of multi- periods' consumption-saving problem. Then, I adopt the central bank's behavior in the monetary economy and leave the future issue.

The market equilibrium is shown formally, because consumers' optimality is solved to be consistent in the results in so far model. If multi-periods model influences on this market equilibrium, in our developed model the proof should be changed. In this paper, the changes of suppositions of price expectation, time structure, and the role of central bank will complicate the proof of the market equilibrium. Hence, their proof problems for the existence of market equilibrium are left as so far subjects.

This paper, on description, by 3 periods model, especially, reveals the problem of consumer's optimality. Naturally, inductively, the model could be extended to n periods but since it becomes to be formal complicates, we explain the 3 periods model. Notations and definitions used in this paper follow mainly to terms of Grandomont [6, 7] and Nishimura [21, 22]

Ⅱ. Notations and Definitions of Model

All consumers have 3 planning periods. Period 1 denotes present, period 2 and period 3 denote successive future. There exist l_1 kinds of perishable consumption goods till period 2 which are available for a consumer at period 1. At period 2, there exist l_2 kinds of the same consumption goods and at period 3, l_2 kinds. Let $l = l_1 + l_2 + l_3$.

Next, the consumer can save a part of income by 2 kinds of financial assets, that is, perpetuities and fiat money. Let money only means of saving and having no interest. On the other hand, at each period perpetuities pay for unit of money per unit of perpetuities to a bearer of perpetuities.

Furthermore, as an economic entity except consumers, there is a central bank which arranges money holdings amount of consumers by trading perpetuities. Our considering problem is basically to show an existence of equilibrium in the l_1+2 markets, and in our paper, the problems of multi-periods' price expectation, the generalization of dynamic programing method on 2 periods model, and the relationship between expected utility hypothesis and time preference theory will be clarified. As below, we describe definitions and assumptions for the explanation of clarifications. They are applied to each consumer, but we omit suffixes to discriminate consumers.

Each consumer has an endowment of consumption $e = (e_1, e_2, e_3) \in R_+^l$ for 3 periods and an initial stock $b_0 = (b_{00}, m_0) \in R_+^2$, where $e_1 \in R_+^{l_1}$, $e_2 \in R_+^{l_2}$, $e_1 \in R_+^{l_3}$, and let each certainly be estimated, respectively. At period 1, $b_{00} \in R_+$ is beginning perpetuities and $m_0 \in R_+$ is beginning balance. m_0 includes interest payment of perpetuities. At the beginning of period 1, each consumer has no debt.

Assumption 1. Let $e \gg 0$, $b_0 \gg 0$.

Let a stream of consumption for 3 periods $x = (x_1, x_2, x_3) \in R_+^l$. Let a stream of assets stock $b = (b_1, b_2, b_3) \in R_+^6$. Market price vectors at period 1 are $p^1 = (p_1, r_1) \in R_+^{l_1} / \{0\}$ and $r^1 = (r_{11}, r_{12})$, where p^1 denotes a price vector of consumption goods, and r^1 an asset price vector. r_{11} is a price of perpetuity and r_{12} is a money price. A market price vector is normalized as follows:

$$p^1 \in \Delta^{l_1+2} = \{ p^1 \in R_+^{l_1+2} \mid \Sigma_{i=1}^{l_1} p_{1i} + \Sigma_{i=1}^{2} r_{1i} = 1 \}.$$

A consumption-savings plan of a consumer is decided at period 1. We call a consumption-savings decision at period 1 an action and denote it as $(x_1, b_1) \in R_+^{l_1+2}$. We call consumption-savings decisions at periods 2 and 3, plans and denote the plan at period 2 as $(x_2, b_2) \in R_+^{l_2+2}$, the plan at period 3 as $(x_3, b_3) \in R_+^{l_3+2}$. The calculation price vectors is $p^2 = (p_2, r_2) \in R_+^{l_2} / \{0\}$ at period 2 and $p^3 = (p_3, r_3) \in R_+^{l_3} / \{0\}$ at period 3. Furthermore, p^2 and p^3 are normalized as $p^2 \in \Delta^{l_2+2} = \{ p^2 \in R_+^{l_2+2} \mid \Sigma_{i=1}^{l_2} p_{2i} + \Sigma_{i=1}^{2} r_{2i} = 1 \}$ and $p^3 \in \Delta^{l_3+2} = \{ p^3 \in R_+^{l_3+2} \mid \Sigma_{i=1}^{l_3} p_{3i} + \Sigma_{i=1}^{2} r_{3i} = 1 \}$.

The consumer decides to choose his action $a_1 = (x_1, b_1)$ under some restricted conditions, his plans for 2 periods $a_2 = (x_2, b_2)$ and $a_2 = (x_3, b_3)$. We formalize this 3 periods temporary general equilibrium model by the extension of Grandmont's 2 periods model.

The complication of the problem followed by multi-periods model will be reduced by multi-variate von Neumann-Morgenstern[1]. If we follow to Grandmont's formalization, the consumer must make a decision under the uncertainty of the future environment for the successive 2 periods. Grandmont assumes that the preferences to the consumption stream satisfies so called expected utility hypothesis[2]. Since Grandmont inherits Patinkin's temporary general equilibrium theory, he insists that money and bonds have no direct utility. Accordingly, under the uncertainty the space of the consequences of consumer's action and plans is limited within the space of consumption stream.

The uncertainty effects on price vectors within the restriction in the decision of consumer's action and plans. Assume that the consumer expects price vector p^2 of period 2 and price vector p^3 of period 3. But consumer's direct preferences to expected price vectors are not assumed. In the case of Grandmont's model, the preferences to expected price vectors are combined with ones to consumption stream and indirectly the price expectation is decided. In multi-periods model, this relationship between preferences to expected price vectors and ones to consumption stream has not been explicitly indicated. As Radner[25] says, in the system of temporal general equilibrium model the price expectation method is not proposed and given exogenously[3].

Along these outline of the formulation on consumer's consumption-savings decision, we begin to consider the 3 periods' model.

Since the consumer faces on the uncertainty in his intertemporal consumption stream decision, he prefers to a probability distribution of consumption stream. Let a set of feasible consequences of consumption stream $(x_1, x_2, x_3) \in C = R_+^{l1} \times R_+^{l2} \times R_+^{l3}$. On product space $\Pi_{i=1}{}^3 M(R_+^{li})$ of set $M(R_+^{li})$ of all probability distributions defined on the space R_+^{li} of the feasible consequences of consumer's action at each period i, the consumer's preferences are represented by a complete preordering $\gtrsim$.

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[1] The extension of Grandmont's model to this direction is studied by Chetty & Dasgupta. I seem that it is more effective method to introduce our utility function than their one to understand the subjects of problem of Grandmont's model.

[2] Grandmont assumes that preferences to consumption stream for 2 periods obey the expected utility hypothesis, as if the uncertainty would exist in period 1, but strictly speaking, the consumer faces on the uncertainty only over the future environment. Hence, according to our formalization of multi-periodization, period 1 is a decision-making under the certainty, and period 2 and period 3 ones under the uncertainty. Then, we implicitly assume that decision-makings in two different environments are compared with the same utility criterion.

[3] Grandmont proposes to replace expected prices with Bayesian statistics. Actually, the proposal has not yet been studied. How to relate between Grandmont's expectation formation and price estimation by rational expectations theorists who introduce it into the general equilibrium theory is not studied. See the introduction by Grandmont[9].

Assumption 2. There exists a von Neumann-Morgenstern utility $u: C \to R$ which is continuous, bounded, such that the mapping $v: \Pi_{i=1}{}^3 M(R_+{}^{li}) \to R$ defined by $v(\mu) = \int_C u \, d\mu$ for any $\mu \in \Pi_{i=1}{}^3 M(R_+{}^{li})$ is a representation of the preferences $\gtrsim$.

Since in Assumption 2, the domain of the mapping v is the product space $\Pi_{i=1}{}^3 M(R_+{}^{li})$, v can have the following form according to Nishimura[21].

$v(\mu) = \Sigma_{i=1}{}^3 \int_{R_+{}^{li}} u_i \, d\mu_i$, for any $\mu = (\mu_1, \mu_2, \mu_3) \in \Pi_{i=1}{}^3 M(R_+{}^{li})$.

Next, We formalize the process of the consumer' decision-making. Since the consumer is a price-taker, at period 1, given a price vector $p^1 \in \Delta^{l1+2}$, he must choose an action a_1 restricted by a subset of an action space $A_1 = R^{l1+2}$, which is denoted by $\beta_1(p^1) = \{ a_1 \in A_1 | \ p^1 \cdot a_1 \leqq p_1 \cdot e_1 + r_1 \cdot b_0 \}$. Now, under the uncertainty, when the future prices p^2 and p^3 realize, the consumer makes plans $a_2 \in A_2 = R_+{}^{l2+2}$ and $a_3 \in A_3 = R_+{}^{l3+2}$. These plans are restricted by subsets of A_2 and A_3 which depend on the chosen action a_1 and the made plan a_2, given the present price p^1 and the future prices p^2 and p^3. These subsets are represented by $\beta_2(a_1, p^2) = \{ a_2 \in A_2 | \ p^2 \cdot a_2 \leqq p_2 \cdot e_2 + r_2 \cdot b_1{}^{\sim} \}$ and $\beta_3(a_1, a_2, p^3) = \{ a_3 \in A_3 | \ p^3 \cdot a_3 \leqq p_3 \cdot e_3 + r_3 \cdot b_2{}^{\sim} \}$, where $b_1{}^{\sim} = (b_1, m_1 + b_1), b_2{}^{\sim} = (b_2, m_2 + b_2)$, and each second term shows to add interest payment to money balance.

In short, the consumer is a price-taker on the decision of action at period 1 and makes plans as a price-taker at period 2 and 3. Then, his plans are denoted by a measurable functions $a_2: \Delta^{l2+2} \to A_2$, that is, $a_2(p^2)$ and $a_3: \Delta^{l3+2} \to A_3$, that is, $a_3(p^3)$, which depend on the future prices p^2 and p^3. Hence, a set of the consumer's action and plans $(a_1, a_2(p^2), a_3(p^3))$ denotes the consumer's choice given the present price, at the realization of future prices p^2 and p^3.

If we assume the expectation formation of future prices p^2 and p^3, then the plans for 2 periods are decided. In the case of Grandmont's expectation function, he takes the form of probability distribution at period 2, conditioned by non-probability variable p^1. But in our case, we must predict 2 periods' prices. In general, the expectation of prices p^2 and p^3 by the consumer at period 2 is a probability distribution on Δ^{l2+2}, and at period 3 the conditional probability distribution on Δ^{l3+2} given the probability variable p^2. Let spaces of all their probability distributions $M(\Delta^{l2+2})$ and $M(\Delta^{l3+2} | B_1(\Delta^{l2+2}))$ [4]. But properties

[4] Given a probability space $(\Delta^{l2+2}, B(\Delta^{l2+2}), \phi_2)$ and σ-field $B_1(\Delta^{l2+2}) \subseteq B(\Delta^{l2+2})$, the conditional probability of the event $G_1 \in B_1$ is any function f with the properties

on the space $M(\Delta^{l3+2} | B_1 (\Delta^{l2+2}))$ of all the conditional probability distribution will request technical special assumptions. Furthermore, to be endowed with weak convergence topology on this space will make the problem of expectation formulation more complicated. We transfer the problem as the subject in the future and assume that the future price at period 3 is independent of the price at period 2. Hence, the space of all their probability distribution at period 3 is $M(\Delta^{l3+2})$.

A set of the consumer's expected prices is defined as follows:

$$\phi : \Delta^{l1+2} \rightarrow M(\Delta^{l2+2}) \times M(\Delta^{l3+2}).$$

Specifically, for any p^1, to take events $G_1 \in B(\Delta^{l2+2})$ and $G_2 \in B(\Delta^{l3+2})$, $\phi(p^1, G_1, G_2)$ is $(\phi_1(p^1, G_1), \phi_2(p^1, G_2))$ and each ϕ_i $(i = 1, 2)$ is a probability imposed on the event G_i given p^1.

Now, we have two a priori beliefs showed by each element of 2 product spaces $\Pi_{i=1}^{3} M(R_+^{li})$ and $M(\Delta^{l2+2}) \times M(\Delta^{l3+2})$. Then, Grandmont insists that the source of the future environment or 'state of natures' implicitly relates to two a priori beliefs. This is one of simplification to explain the temporal general equilibrium theory under the uncertainty and he assumes that all consumers have a priori beliefs about the future environment. Of course, generally, it cannot be said that the realization of consumption in the future must be consistent with the expected prices at that time. We transfer the problem as the subject in the future (See footnote 3) and follow to Grandmont's assumption.

The consumer has the following mapping γ.

$\gamma : \Delta^{l1+2} \times A_1 \times a_2(\Delta^{l2+2}) \times a_3(\Delta^{l3+2}) \rightarrow C$, where $a_2(\Delta^{l2+2}) \subset A_2$, $a_3(\Delta^{l3+2}) \subset A_3$.

Suppose that this mapping γ is measurable, linear. The mapping γ means that if given price vector p^1, the choice $(a_1, a_2(p^2), a_3(p^3))$ is decided by the realization of price vectors p^2 and p^3, then a consequence of consumption stream (x_1, x_2, x_3) is obtained. Since γ is assumed to be linear, each element of the consequence is consistent with each consumption element of the choice.

A random varable γ induces a probability distribution on $C = R_+^{l1} \times R_+^{l2} \times R_+^{l3}$. Since γ is linear, for every event $S = S_1 \times S_2 \times S_3 \in B(\Pi R^{li})$, $\gamma^{-1}(S) = \gamma_1^{-1}(S) \times \gamma_2^{-1}(S) \in B(\Delta^{l2+2} \times \Delta^{l3+2})$. Since probability distributions on Δ^{l2+2} and Δ^{l3+2} are assumed mutually independent, for any p^1, the product distribution $\phi_1(p^1, \gamma_1^{-1}(S)) \times \phi_2(p^1, \gamma_2^{-1}(S))$ correspond to the set of expected price probability distributions (ϕ_1, ϕ_2). This shows

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that (1) f is B measurable and (2) for any $G_1 \in B_1$, $\int_{G_1} f d\phi_2 = \phi_2(p^1, G \cap G_1)$, where f depends on G and is denoted by $\phi_2(p^1, G | B_1)$. This is based on definitions of Parthasarathy[24, Ch. 5].

that a probability distribution on C is induced by a measurable mapping γ. Therefore, let this induced probability distribution $\mu_{p1, a1, a2(p2), a3(p3)}$. We define

$\mu_{p1, a1, a2(p2), a3(p3)} = \phi_1(p^1, \gamma_1^{-1}(S)) \times \phi_2(p^1, \gamma_2^{-1}(S))$.

Then, expected utility of $\mu_{p1, a1, a2(p2), a3(p3)}$ is defined by Assumption 2 as follows:

$$\int_{\Delta^{l2+2} \times \Delta^{l3+2}} u(p^1, a_1, a_2(\cdot), a_3(\cdot))\, d\phi_1 \times d\phi_2 = \int_C u\, d\mu_{p1, a1, a2(p2), a3(p3)}. \qquad (1)$$

Furthermore, using a multi-variate von Neumann-Morgenstern utility function, this expected utility is represented as follows[5]:

$$\int_{\Delta^{l2+2}} u_1\, d\phi_1 + \int_{\Delta^{l3+2}} u_2\, d\phi_2 = \Sigma_{i=1}^{3} \int_{R_+^{li}} u_i\, d\mu_i{}^{6}. \qquad (2)$$

By above results, we obtained the utility function in order to the decision-making of 3 periods' model. Finally, we describe the assumptions on expectation function ϕ_i.

Assumption 3. (1) Any von Neumann-Morgenstern utility u is concave and monotone.

(2) Each expectation function ϕ_i is continuous.

(3) For every $p^1 \in \Delta^{l1+2}$, $\phi_i(p^1)$ assigns probability one to the set of $p^2 \in \Delta^{l2+2}$, $p^3 \in \Delta^{l3+2}$ such that money price $r_i > 0$ (i=2,3).

The meanings of these assumptions are that (1) the consumer behaves as a risk-averter, (2) it holds the continuity of the utility function, and (3) even if money price is 0, all consumers expect that money price is positive at some probability in the future.

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5 In this manner, preference ordering of $\gtrsim$ is induced on the space $M(\Delta^{l2+2}) \times M(\Delta^{l3+2})$ and the product measure is additively decomposed, because preference ordering as same as $\gtrsim$ is assumed on $M(\Delta^{l2+2}) \times M(\Delta^{l3+2})$, in Grandmont[6]. Suppose a complete preordering $\gtrsim^c$ on the space C of consequences. For any price , γ_1^{-1} induces a complete preordering $\gtrsim^c{}_{p1}$ on $A = A_1 \times a_2(\Delta^{l2+2}) \times a_3(\Delta^{l3+2})$. Accordingly, the following relationship is assumed.

For any $p^1 \in \Delta^{l1+2}$, when $a_1, a_2 \in A$, $x_1, x_2 \in C$, $a_1 \gtrsim^c{}_{p1} a_2$ if and only $x_1 \gtrsim^c x_2$. This relation is important to show the monotony of expected utility function.

6 Each u_i is a function whose reference element of the origin 0 is fixed as constant, by Nishimura[22]. That is, let a reference element $(x_1^0, x_2^0, x_3^0) \in C$, $u_1 = u_1(x_1, x_2^0, x_3^0)$, $u_2 = u_2(x_1^0, x_2, x_3^0)$, $u_3 = u_3(x_1^0, x_2^0, x_3)$. Suppose equivalence relation of footnote 5. A reference element on the space A of actions and plans, given a_1 for any p^1, (p^1, a_1, a_2^0, a_3^0) corresponds to the reference point (x_1^0, x_2^0, x_3^0) on C by a mapping γ. The first term x_1^0 corresponds firstly to a_1^0, next, to given a_1 by a positive linear transformation (This point refers to Nishimura[22]). Therefore, each u_i is $u_1 = u_1(\gamma(x_1, x_2^0, x_3^0))$, $u_2 = u_2(\gamma(x_1^0, x_2, x_3^0))$, $u_3 = u_3(\gamma(x_1^0, x_2^0, x_3))$.

Ⅲ. Consumer’ s Decision-making Problem

The consumer chooses an action a_1 at period 1 which belongs to the set of budget constraint $\beta_1(p^1)$, and makes plans $a_2(p^2) = (x_2, b_2)$and $a_3(p^3) = (x_3, b_3)$ which belong to the sets of budget constraint $\beta_2(a_1, p^2)$, $\beta_3(a_1, a_2, p^3)$, respectively. Then, the problem is as follows:

$$\max_{\{a_1,\, a_2,\, a_3\}} \int_{\Delta^{l2+2} \times \Delta^{l3+2}} u(\gamma(p^1, a_1, a_2(\cdot), a_3(\cdot)))\, d\phi_1(p^1) \times d\phi_2(p^2), \tag{3}$$

$$\begin{aligned} \text{subject to } \beta_1(p^1) &= \{ a_1 \in A_1 \mid p^1 \cdot a_1 \leqq p_1 \cdot e_1 + r_1 \cdot b_0 \}, \\ \beta_2(a_1, p^2) &= \{ a_2 \in A_2 \mid p^2 \cdot a_2 \leqq p_2 \cdot e_2 + r_2 \cdot b_1^{\sim} \}, \\ \beta_3(a_1, a_2, p^3) &= \{ a_3 \in A_3 \mid p^3 \cdot a_3 \leqq p_3 \cdot e_3 + r_3 \cdot b_2^{\sim} \}. \end{aligned}$$

Furthermore, this expected utility maximization problem is transformed to the following form without changing the budget constraints.

$$\max_{\{a_1,\, a_2,\, a_3\}} \int_{\Delta^{l2+2}} u_1\, d\phi_1(p^1) + \int_{\Delta^{l3+2}} u_2\, d\phi_2(p^2). \tag{4}$$

By transforming Problem (3) to Problem (4), we can easily solve it to use well-known backward Dynamic Programming. The right hand side of the equation (2) shows the additive form of utility functions at each period, often using in the time preferences theory, and the left hand side of the equation (2) describes that it is transformed into utility functions for 2 periods. This ensures theoretical consistency in the sense that when we extend 2 periods’ model to 3 periods’, according to the equivalence relation of preference orderings, von Neumann-Morgenstern utility naturally is applied to the time preferences theory.

Now, we shall show the solution of Problem (4) by the backward Dynamic Programming. The consumer takes an action a_1 and a plan a_2 at period 3, and faces on price vector p^3 at period 3. Problem (4) is as follows:

$$\max_{\{a_3\}} u_2(\gamma(p^1, a_1, a_2^0, a_3(p^3))) + \int_{\Delta^{l2+2}} u_1(\gamma(p^1, a_1, a_2, a_3^0))\, d\phi_1(p^1), \tag{5}$$

subject to $\beta_3(a_1, a_2, p^3)$.

If there exists a solution of Problem (4), we denote an optimal plan by $a_3{}^*(p^3)$. To substitutes $a_3{}^*$ for the utility function $u_2(\gamma(p^1, a_1, a_2{}^0, a_3(p^3)))$, we obtain the optimal value given p^1 and a_1 as follows:

$u_2{}^*(a_1, p^1, p^3) = u_2(\gamma(p^1, a_1, a_2{}^0, a_3{}^*(p^3)))$.

Then, since the consumer takes an action a_1, and faces on price vector $p^2 \in \Delta^{l_2+2}$at period 2, the problem at period 2 is the following.

$$\max_{\{a_2\}} \quad u_1(\gamma(p^1, a_1, a_2(p^2), a_3{}^0)), \tag{6}$$

subject to $\beta_2(a_1, p^2)$.

If there exists a solution of Problem (6), we denote an optimal plan by $a_2{}^*(p^2)$ and to substitute $a_2{}^*$ for the utility function u_1, given p^1 and a_1, we obtain as follows:

$u_1{}^*(a_1, p^1, p^2) = u_1(\gamma(p^1, a_1, a_2{}^*, a_3{}^0(p^3)))$.

Finally, the problem at period 1,

$$\max_{\{a_1\}} \qquad \int_{\Delta^{l_2+2}} u_1{}^*(a_1, p^1, \cdot)\, d\phi_1(p^1) + \int_{\Delta^{l_3+2}} u_2{}^*(a_1, p^1, \cdot)\, d\phi_2(p^2).$$

subject to $\beta_1(p^1)$.

Let $v(a_1, p^1) = \int u_1{}^* d\phi_1(p^1) + \int u_2{}^* \, d\phi_2(p^2)$. The function $v(a_1, p^1)$ is called indirect utility function of financial assets or simply expected utility.

Above, we could have the solution of Dynamic Programming of Problem (4). In the case of n periods, it will be easy inductively to extend the solution.

In Grandmont's theory, the above expected utility $v(a_1, p^1)$ corresponds to the usual utility function in the general equilibrium theory. When the properties of expected utility v are decided, the individual demand correspondences are defined, and we can prove the existence of market equilibrium to define the aggregate excess demand correspondence by the summation of the individual demand correspondences. Though in our 3 periods' model, we use the multi-variable von Neumann-Morgenstern utility, it is proved that the properties of v are the same as Grandmont's utility.

Proposition 1. Expected utility function $v(a_1, p^1)$ is continuous, 凹, and strictly monotone, for every $p^1 \in \Delta^{l_1+2}$and $a_1 \in \beta_1(p^1)$.

Proof. Since u_1 and u_2 are 凹 by assumption 3, the sum of 凹 functions is 凹. The concavity of function v can apply Lemma 7.2 of Sondermann [26] to our case. The strictly monotony is evident since u_1 and u_2 are 凹 by assumption 3. We prove only that v is continuous. Taking $(a_1{}^0, p^{10}) \in A_1 \times \Delta^{n+2}$, consider sequences $\{(a_1{}^j, p^{1j})\}$ which converge to this point. Now, functions $u_1{}^*(a_1{}^j, p^{1j}, \cdot)$ and $u_2{}^*(a_1{}^j, p^{1j}, \cdot)$ are uniformly bounded and they converge continuously to $u_1{}^*(a_1{}^0, p^{10}, \cdot)$ and $u_2{}^*(a_1{}^0, p^{10}, \cdot)$. On the other hand, since each expectation function ϕ_i is continuous, sequences $\{\phi_1(p^{1j})\}$ and $\{\phi_2(p^{1j})\}$ weakly converge to $\phi_1(p^{10})\}$ and $\phi_2(p^{10})$. Therefore, lim $v(a_1{}^j, p^{1j}) = v(a_1{}^0, p^{10})$[7].

Ⅳ. Central Bank

About the behavior of central bank, we obey assumptions of Grandmont and Laroque [11]. In our 3 periods' model, we can introduce the short term or the long term maturity of bonds except perpetuity. We can introduce the term structure among rates of interest by bonds with maturities and central bank controls money supply over multi-periods, with the operation among the short term interest rates and the long term interest rates. However, why central bank requests the term structure of rates of interest should seek the reason for the consumers' financial behavior in our simple model. As described in the time preference theory, mainly, according to the variation of the consumer's income pattern over time the consumer will hold or sell bonds with the term structure, or plan to borrow the consumption loan or repay debt in order to optimize the expected consumption at each period. Then, the consumer has the long term plan optimized by the life cycle theory.

As another theoretical justification to consider an economy with the term structure there is the system that young generation has central bank accept bonds with the term structure in the Samuelson's consumption loan model with overlapping generations[8]. Then, central bank holds multi-periods' portfolios of consumption loan and it is justified that central bank behaves to consider the term structure of interest.

In the former case of the life cycle theory, central bank will issue bonds with the

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7 Since [6. Section 2, 3] is extended to multi-variate von Neumann-Morgenstern utility function by Nishimura [21], the same theorem as Grandmont [6. Section 5, Theorem A. 3] is obtained even in our case. It is obvious that it is applied to the proof of the continuity of our v function.

8 Our 3 periods' model can easily convert to the consumption loan model. Then, the role of central bank is to issue fiat money with which young consumers buy bond and old consumers repay debt. 2 periods' model of Grandmont and Laroque [10] has 2 generations of the young and the old, since in our model, overlapping generations are naturally extended to 3 generations, the consumption loan model can have the term structure.

term structure. The central bank has, as the behavior norm without for-profit purposes, the purpose to prevent the collapse of credit order at each period, that is, consumers' bankruptcy in the whole economy[9]. Therefore, under the uncertainty, the central bank must expect rather exactly the consumers' bonds demand at each period. But Grandmont's price expectation formation does not give him the concrete method to estimate the bonds prices, because unlike the consumer's price expectation, central bank's internal expectation of bonds demand is based on the decision on bonds issues with the term structure and the issues conditions are decided according to the central bank's prediction of the possibility of the consumers' bankruptcies. In that sense, Grandmont's price expectation is exogenous in the formulation of central bank's action and it is not evident how the short term interest and the long term interest are decided. By the above reason, our model of the extension of Grndmont's to 3 periods obeys Grandmont and Laroque [10] without the term structure of bonds. However, if we change price expectation theory, there is the thorough possibility that the attractive theme of multi-periods could be treated. They are left as future subject.

The action of central bank is to conduct open market operations and pay for interest at period 1. Thus, the central bank chooses an action $a = (x, b, m) \in R^{n+2}$, where x=0, and this denotes net supply of consumption of the bank. b and m are net supply of bond and money, respectively. Given $p^1 = (p_1, r_1) \in \Delta^{n+2}$, $r_1 = (r_{11}, r_{12})$, we assume that the rate of interest $r = r_{12} / r_{11} > 0$. Then, b and m must be related by the following accounting identity. When n persons meet in the markets,

$m = -b / r + \Sigma_{i=1}{}^{n} b_{00}{}^{i}$, where $b_{00}{}^{i}$ is the i consumer's stock of bond at the beginning of period 1. This identity shows that net supply of money equals to money issue for the purchase of bond in the open market and interest payment of outstanding bond.

The monetary policy is defined by the bank supply correspondence, that is, the subset $\eta(p^1)$(the empty set will be possible.) in the feasible set of the bank action $\beta(p^1) = \{ a \in R^{n+2} |\ x=0, m = -b / r + \Sigma_{i=1}{}^{n} b_{00}{}^{i} \}$. The monetary policy is divided into the case to fix the rate $r>0$ of perpetuity at any level, the case to fix the issue amount of perpetuity, and the case to fix the amount of money. We consider only the first policy. If $r = r_{12} / r_{11} > 0$, let $\eta^{r}(p^1) = \beta(p^1)$, otherwise, $\eta^{r}(p^1)$ empty set.

V. Market Equilibrium

[9] In period 1, each consumer makes his plan to prevent from his bankruptcy when he takes his action and chooses his plan. But Bliss [1] insisted that financial institutes took some role to rescue the bankruptcy.

Again, back to the section Ⅲ, consider the consumer's optimal action at period 1. We shows the existence of the temporal equilibrium at period 1. In this step, our 3 periods' model, compared with Grandmont's 2 periods' model does not differ from Grandmont's Propositions. It results from that bond issued by the central bank has no term structure. Hence, the theoretical framework of Grandmont's market equilibrium is applied to our model.

At period 1, the ith consumer chooses an action from the set of budget constraint $\beta^i(p^1)$. For any p^1, $\beta^i(p^1)$ is nonempty, 凸, compact-valued, and upper hemi-continuous. Then, the demand correspondence of the ith consumer $\xi^i(p^1)$ is defined by

$$\xi^i(p^1) = \{\, a_1 \in \beta^i(p^1) \mid v(a_1, p^1) \geqq v(a_1', p^1), \text{ for all } a_1' \in \beta^i(p^1)\}$$

Let D open 凸 subset of Δ^{n+2}. Then, the below propositions are proved true[10].

Proposition 2. $\xi^i(p^1)$ is nonempty and 凸 on D, and compact-valued, and upper hemi-continuous. For every $p^1 \in D$, $p^1 \cdot \xi^i(p^1) = p^1 \cdot (e_1^i, b_0^i)$.

Proposition 3. Take any sequence $\{p^{1j}\}$ such that $p^{1j} \to p^{1-} \in \partial D$. Consider any sequence $\{a_1^j\}$ such that $a_1^j \in \xi^i(p^{1j})$. Then, $\| a_1^j \| \to +\infty$.

Suppose that there are n consumers in the markets. The ith consumer's excess demand correspondence is defined by

$\zeta^i(p^1) = \xi^i(p^1) - (e_1^i, b_0^i)$.

Next, the aggregate excess demand correspondence is defined by

$\zeta(p^1) = \Sigma_{i=1}^{n} \zeta^i(p^1)$.

Finally, the market equilibrium to correspond to the monetary policy η^r is defined by the price system $p^1 \in D$ such that $0 = \zeta(p^1) - \eta^r(p^1)$. Then, the following Theorem is established.

Theorem 1. Under Assumptions, $\Sigma_{i=1}^{n}(e_1^i, b_0^i) \gg 0$. Then, if $r = 0$, there exists no equilibrium correspondent to η^r. if $r > 0$, there exists an equilibrium correspondent to η^r.

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10 Proposition 2 and Proposition 3 are proved by Grandmont [7, Propositions 4. 1, 4. 2]. Theorem 1 Is by Grandmont and laroque [11, Theorem 1].

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